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Case study in bivariate Hermite interpolation

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Abstract

In this article we investigate the minimal dimension of a subspace of $C^1(\mathbb{R}^2)$ needed to interpolate an arbitrary function and some of its prescribed partial derivatives at two arbitrary points. The subspace in question may depend on the derivatives, but not on the location of the points. Several results of this type are known for Lagrange interpolation. As far as I know, this is the first such study for Hermite Interpolation.

1. Introduction and $\{\delta_u, \delta_v\}$-interpolating case

Multivariate Hermite Interpolation has been studied extensively in the last 30 years. Excellent surveys on recent accomplishments can be found in [4,8,9]. Naturally, most of the questions are centered around the similarities and differences from the univariate case. The most apparent difference is the lack of unicity for Hermite interpolation in the multivariate case. Hence there are studies of those configurations of points and derivatives for which the Hermite interpolation problem is uniquely solvable (correct, proper, well defined....) in a given space, usually the space of polynomials of a given degree. We refer to [5,6] as examples of such studies. There is another approach (cf. [2,10]) where one starts with arbitrary Hermite data and designs the space to suit the needs. This article is different. While this study still starts from the lack of correctness in Hermite interpolation, we are looking for spaces for which a certain Hermite interpolation problem is solvable for any configurations of interpolation points. Hence the dimension of these spaces may, by necessity, be larger

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then the number of data and the uniqueness is not an option. The second difference is that we are contrasting multivariate Hermite interpolation with multivariate as well as univariate Lagrange interpolation. In some cases we show that multivariate Hermite problem for arbitrary configuration of points may be closer to the univariate problem than the similar problem for Lagrange interpolation. To be precise, we investigate the minimal dimension of a subspace of $C^1(\mathbb{R}^2)$ needed to interpolate an arbitrary functions and some of its prescribed partial derivatives at two arbitrary points. The subspace in question may depend on the derivatives, but not on the location of the points. For Lagrange interpolation several results of this type are known (cf. [3,12–15]). As far as I know, this is the first such study for Hermite (Lagrange) Interpolation.

Let $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\} \subset (C^1(\mathbb{R}^2))^*$ be a finite collection of functionals defined on $C^1(\mathbb{R}^2)$. Let $\Phi \subset C^1(\mathbb{R}^2)$ be a finite-dimensional subspace. We say that Φ *is* Λ*-interpolating* if for any sequence of scalars $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{R}$, there exists a function $f \in \Phi$ such that $\lambda_j(f) = \alpha_j$ for all $j = 1, 2, \ldots, n$.

Let $f_1, f_2, \ldots, f_m$ be a basis for Φ. Define the $n \times m$ matrix $\tilde{\Phi} := [\lambda_j(f_k)]$. Clearly, the space Φ is Λ-interpolating if and only if

$$rank\,\tilde{\Phi} = rank[\lambda_j(f_k)] = n.$$

Observe that the matrix $\tilde{\Phi}$ depends on the basis $\{f_j\}$, but $rank\,\tilde{\Phi}$ is independent of the choice of the basis. It is also obvious that if Φ is $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$-interpolating, then $m := \dim \Phi \geqslant n$.

As an example consider the case of Lagrange interpolation: $\Lambda = \{\delta_u, \delta_v\}$ where $u, v \in \mathbb{R}^2$ and $\delta_w \in (C^1(\mathbb{R}^2))^*$ is defined by $\delta_w(f) = f(w)$. Let $u = (a, b)$ and $v = (c, d)$.

If $u = v$, then there are no Λ-interpolating spaces. If $u \neq v$ and $a \neq c$ then the linear span Φ_1 of the functions $f_1(x, y) := 1$ and $f_2(x, y) := x$ is $\{\delta_u, \delta_v\}$-interpolating. In fact the space Φ_1 is a $\{\delta_u, \delta_v\}$-interpolating space of the least possible dimension ($\dim \Phi_1 = 2$). Similarly if $u \neq v$ and $b \neq d$ then $span[1, y]$ is $\{\delta_u, \delta_v\}$-interpolating. Hence the three-dimensional space $\Phi := span[1, x, y]$ is $\{\delta_u, \delta_v\}$-interpolating for any $u \neq v \in \mathbb{R}^2$. The natural question to ask is whether there exists a two-dimensional space $\Phi \subset C^1(\mathbb{R}^2)$ which is simultaneously $\{\delta_u, \delta_v\}$-interpolating for any $u \neq v \in \mathbb{R}^2$? The answer is given by the famous "Mairhuber Theorem" (cf. [7]):

Theorem 1. *For any two-dimensional subspace $\Phi = span[f_1, f_2] \subset C(\mathbb{R}^2)$ there exists a pair of distinct points $u, v \in \mathbb{R}^2$ such that the space Φ does not interpolate at these points.*

Since we will use the Mairhuber argument elsewhere in this paper, (and since the idea is very cute) let us reproduce it.

Proof of Theorem 1. Let $\Phi = span[f_1, f_2]$. Position two points u, v on diametrically opposite ends of a circle and consider the matrix

$$\tilde{\Phi}[u, v] = \begin{bmatrix} f_1(u) & f_2(u) \\ f_1(v) & f_2(v) \end{bmatrix}.$$

As we rotate the diameter, the points u and v switch positions and hence $\det \tilde{\Phi}[u, v]$ changes sign. By the intermediate value theorem, there exists a pair u, v such that $\det \tilde{\Phi}$ $[u, v] = 0$; hence Φ is not interpolating at these points. $\square$

This theorem together with the preceding remarks settles the first case of Hermite interpolation:

The minimal dimension of a space Φ that interpolates functionals $\{\delta_u, \delta_v\}$ for any $u \neq v \in \mathbb{R}^2$ is three, and $\tilde{\Phi} := span[1, x, y]$ is such a space.

The rest of this paper is dedicated to similar questions with the collection of functionals Λ consisting not only of point-evaluations δ_u, δ_v but also of the derivatives at those points: $\delta_w \circ \frac{\partial}{\partial v}$ where $0 \neq v \in \mathbb{R}^2$ is a given direction.

2. Case 2: $\{\delta_u, \delta_v, \delta_u \circ \frac{\partial}{\partial v}\}$-Hermite interpolation

The four-dimensional space $\Phi := span[1, x, y, x^2 + y^2]$ is $\{\delta_u, \delta_v, \delta_u \circ \frac{\partial}{\partial v}\}$-interpolating for any $u \neq v \in \mathbb{R}^2$ and for any $v \in \mathbb{R}^2 \backslash \{0\}$.

Given any three-dimensional space $\Phi \subset C^1(\mathbb{R}^2)$ and any fixed direction $v \in \mathbb{R}^2$ there exist $u \neq v \in \mathbb{R}^2$ such that Φ is not $\{\delta_u, \delta_v, \delta_u \circ \frac{\partial}{\partial v}\}$-interpolating.

The first claim is easy to verify directly. It also follows from the theorem in the next section.

To prove the second statement we need to show that given any three functions f_1, f_2, $f_3 \in C^1(\mathbb{R}^2)$ and any direction $v \in \mathbb{R}^2$ the determinant

$$\begin{vmatrix} f_1(u) & f_2(u) & f_3(u) \\ f_1(v) & f_2(v) & f_3(v) \\ \frac{\partial f_1}{\partial v}(u) & \frac{\partial f_2}{\partial v}(u) & \frac{\partial f_3}{\partial v}(u) \end{vmatrix} = 0 \quad \text{for some } u \neq v \in \mathbb{R}^2.$$

This will follow as a corollary from the next, more general, topological theorem, where the vector-valued function $G(u) := (\frac{\partial f_1}{\partial v}(u), \frac{\partial f_2}{\partial v}(u), \frac{\partial f_3}{\partial v}(u))$ which formally speaking depends on the function $F(u) := (f_1(u), f_2(u), f_3(u))$ is replaced with an arbitrary vector-valued function $G : \mathbb{R}^2 \to \mathbb{R}^3$.

Theorem 2. *For every six continuous functions $f_1, f_2, f_3, g_1, g_2, g_3 : \mathbb{R}^2 \to \mathbb{R}$ there exist $u \neq v \in \mathbb{R}^2$ such that the 3×3 determinant*

$$\begin{vmatrix} f_1(v) & f_2(v) & f_3(v) \\ f_1(u) & f_2(u) & f_3(u) \\ g_1(u) & g_2(u) & g_3(u) \end{vmatrix} = 0.$$

Proof. Without loss of generality we can assume that the function $f_3 \equiv 1$ in some neighborhood U of zero. Indeed if $(f_1(0), f_2(0), f_3(0)) = 0$ then the theorem is obvious. If one of the components, say $f_3(0)$ is different from zero, we can divide the first and second rows by $f_3(v)$ and $f_3(u)$, respectively.

To prove the theorem, we assume by way of contradiction that the determinant

$$\begin{vmatrix} f_1(v) & f_2(v) & 1 \\ f_1(u) & f_2(u) & 1 \\ g_1(u) & g_2(u) & g_3(u) \end{vmatrix} \neq 0 \quad \text{for all } u \neq v \in U. \tag{2.1}$$

To this end consider a map $F : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $F(u) = (f_1(u), f_2(u), 1)$. It follows from assumption (2.1) that F is injective and the image $F(\mathbb{R}^2)$ is a subset of the plane $\{(x, y, z) : z = 1\}$. Let $C \subset U$ be a circle centered at 0. By the Jordan curve theorem we conclude that the curve $F(C)$ divides the plane $z = 1$ into two components: a bounded component B and an unbounded component D with $F(C)$ being the boundary common to both. Moreover $F(0) \in B$, since 0 belongs to the disk bounded by C. Consider now the plane $P(0) := span[F(0), (g_1(0), g_2(0), g_3(0))]$. It follows from (2.1) that $P(0)$ is indeed a two-dimensional plane that passes through the origin, and is not parallel to the plane $z = 1$. Hence the intersection of the two planes is a straight line $l = P(0) \cap \{z = 1\}$. The line l contains the point $F(0) \in B$ and a point $w \in D$, since the line cannot belong to the bounded component B. Since the regions B and D are disconnected, it follows that there exists a point $w_1 \in l \cap F(C)$ and hence there exist $v \neq 0$ in $\mathbb{R}^2$ such that $F(v) = w_1 \in l \subset P(0)$. That means that the vectors $F(0)$, $F(v)$ and $(g_1(0), g_2(0), g_3(0))$ belong to the same plane $P(0)$, which contradicts (2.1). □

3. Case 3: Interpolation with two derivatives

In this section we examine the following three subcases:

(1) $\{\delta_u, \delta_v, \delta_u \circ \frac{\partial}{\partial \nu}, \delta_u \circ \frac{\partial}{\partial \mu}\}$-*interpolating* with ν and μ linearly independent directions in $\mathbb{R}^2$
(2) $\{\delta_u, \delta_v, \delta_u \circ \frac{\partial}{\partial \nu}, \delta_v \circ \frac{\partial}{\partial \mu}\}$-*interpolating* with ν and μ linearly independent directions in $\mathbb{R}^2$
(3) $\{\delta_u, \delta_v, \delta_u \circ \frac{\partial}{\partial \nu}, \delta_v \circ \frac{\partial}{\partial \mu}\}$-*interpolating* with $\nu = \mu$

Surprisingly, in all these cases the minimal dimension of the interpolation subspace is four. Unlike the previous case, the two-dimensional nature of the problem does not increase the dimension of the interpolation spaces.

Unless otherwise specified, we will use coordinate notations for the points and the derivatives as follows:

$$u = (a, b), \quad v = (c, d), \quad \nu = (\alpha, \beta) \quad \text{and} \quad \mu = (\gamma, \delta). \tag{3.1}$$

Theorem 3. *Let $\Lambda = \{\delta_u, \delta_v, \delta_u \circ \frac{\partial}{\partial \nu}, \delta_u \circ \frac{\partial}{\partial \mu}\}$. The four-dimensional space $\Phi := span$ $[1, x, y, x^2 + y^2]$ is Λ-interpolating for any $u \neq v \in \mathbb{R}^2$ and for any linearly independent directions ν and μ* in $\mathbb{R}^2$.

Proof. By direct computation, the associated determinant

$$\begin{vmatrix} 1 & a & b & a^2+b^2 \\ 1 & c & d & c^2+d^2 \\ 0 & \alpha & \beta & 2a\alpha+2b\beta \\ 0 & \gamma & \delta & 2a\gamma+2b\delta \end{vmatrix} = -((a-c)^2+(b-d)^2)\,(\beta\gamma-\alpha\delta)\,.$$

Since v and μ are linearly independent, the quantity $(\beta\gamma-\alpha\delta)\neq 0$. Hence this determinant is zero iff $u=v$. $\square$

Theorem 4. *Given two linearly independent directions v and μ in $\mathbb{R}^2$ there exists a four-dimensional space Φ which interpolates $\{\delta_u, \delta_v, \delta_u \circ \frac{\partial}{\partial v}, \delta_v \circ \frac{\partial}{\partial \mu}\}$ for any $u \neq v \in \mathbb{R}^2$.*

Proof. First, consider the case $v=(1,0)$ and $\mu=(0,1)$ and the space $\Phi := span[1, x, y, x^2-y^2]$. Once again, direct computation of the associated determinant yields

$$\begin{vmatrix} 1 & a & b & a^2-b^2 \\ 1 & c & d & c^2-d^2 \\ 0 & 1 & 0 & 2a \\ 0 & 0 & 1 & -2d \end{vmatrix} = (a-c)^2+(b-d)^2.$$

Using a linear change of variables we conclude that the space

$$\Phi := span[1, x, y, (\langle v, (x,y)\rangle)^2 - (\langle \mu, (x,y)\rangle)^2]$$

is $\{\delta_u, \delta_v, \delta_u \circ \frac{\partial}{\partial v}, \delta_v \circ \frac{\partial}{\partial \mu}\}$-interpolating for any $u \neq v \in \mathbb{R}^2$. $\square$

The last subcase is a little more delicate.

Theorem 5. *Given a direction $v \in \mathbb{R}^2\backslash\{0\}$, there exists a four-dimensional subspace Φ that interpolates functionals $\{\delta_u, \delta_v, \delta_u \circ \frac{\partial}{\partial v}, \delta_v \circ \frac{\partial}{\partial v}\}$ for any $u \neq v \in \mathbb{R}^2$.*

Proof. We again assume that $v=(1,0)$. This time the desired space $\Phi = span[1, x, x^2+y, x^3+3xy]$. Indeed

$$\begin{vmatrix} 1 & a & a^2+b & a^3+3ab \\ 1 & c & c^2+d & c^3+3cd \\ 0 & 1 & 2a & 3a^2+3b \\ 0 & 1 & 2c & 3c^2+3d \end{vmatrix} = -(a-c)^4 - 3(b-d)^2.$$

The general direction case follows by linear change of variables. If $v=(\alpha,\beta)\neq 0$, choose

$$X = \alpha x + \beta y, \quad Y = \alpha y - \beta x.$$

The interpolating space is

$$\Phi = span[1, X, X^2+Y, X^3+3XY]. \qquad \square$$

Remark 6. It is interesting to note that none of the spaces presented in the last three theorems is interpolating for any other set of functionals considered in this section.

4. Case 4: Interpolation with three derivatives

In this section we deal with spaces that interpolate the functionals

$$\Lambda = \left\{ \delta_u, \delta_v, \delta_u \circ \frac{\partial}{\partial v_1}, \delta_u \circ \frac{\partial}{\partial v_2}, \delta_v \circ \frac{\partial}{\partial \mu} \right\}.$$

Using linear change of variables, we can restrict our considerations to the collection

$$\Lambda := \left\{ \delta_u, \delta_v, \delta_u \circ \frac{\partial}{\partial x}, \delta_u \circ \frac{\partial}{\partial y}, \delta_v \circ \frac{\partial}{\partial x} \right\}.$$

Proposition 7. *The six-dimensional subspace* $\Phi := span[1, x, y, xy, x^2 - y^2, x^3 - 3xy^2]$ *is* Λ*-interpolating for any* $u \neq v \in \mathbb{R}^2$.

Proof. We wish to show that

$$rank \begin{bmatrix} 1 & a & b & ab & a^2-b^2 & a^3-3ab^2 \\ 1 & c & d & cd & c^2-d^2 & c^3-3cd^2 \\ 0 & 1 & 0 & b & 2a & 3a^2-3b^2 \\ 0 & 0 & 1 & a & -2b & -6ab \\ 0 & 1 & 0 & d & 2c & 3c^2-3d^2 \end{bmatrix} = 5\,. \tag{4.1}$$

Deleting the last column, and evaluating the remaining determinant we obtain:

$$\begin{vmatrix} 1 & a & b & ab & a^2-b^2 \\ 1 & c & d & cd & c^2-d^2 \\ 0 & 1 & 0 & b & 2a \\ 0 & 0 & 1 & a & -2b \\ 0 & 1 & 0 & d & 2c \end{vmatrix} = (d-b)\,((b-d)^2 + (a-c)^2)$$

which is equal to zero if and only if $d = b$.

Setting $d = b$ in matrix (4.1), and deleting the fourth column we have

$$\begin{vmatrix} 1 & a & b & a^2-b^2 & a^3-3ab^2 \\ 1 & c & b & c^2-b^2 & c^3-3cb^2 \\ 0 & 1 & 0 & 2a & 3a^2-3b^2 \\ 0 & 0 & 1 & -2b & -6ab \\ 0 & 1 & 0 & 2c & 3c^2-3b^2 \end{vmatrix} = a^4 - 4ac^3 + 6c^2a^2 - 4ca^3 + c^4 = (a-c)^4$$

which proves the desired result. □

Conjecture 8. *For any five-dimensional subspace* $\Phi \subset C^1(\mathbb{R}^2)$ *there exist points* $u \neq v \in \mathbb{R}^2$ *such that* Φ *is not* Λ*-interpolating.*

In support of this conjecture we offer the following "claim", for lack of a better term.

Claim 9. *Let* $\Phi = span[p_1, p_2, p_3, p_4, p_5]$ *be the span of five polynomials of degree at most three. Then* Φ *is not* Λ*-interpolating.*

Proof. Let $P : \mathbb{R}^2 \to \mathbb{R}^5$ be a mapping defined by

$$P(u) = (p_1(u), p_2(u), p_3(u), p_4(u), p_5(u)), \text{ where } p_k(u) = p_k(x, y) = \sum_{i+j \leqslant 3} a_{i,j}^{(k)} x^i y^j$$

and let $u = (a, b)$ and $v = (c, d)$. To set up a contradiction we assume that the determinant

$$\begin{vmatrix} - & P(a,b) & - \\ - & P(c,d) & - \\ - & \frac{\partial}{\partial x} P(c,d) & - \\ - & \frac{\partial}{\partial y} P(c,d) & - \\ - & \frac{\partial}{\partial x} P(a,b) & - \end{vmatrix} \neq 0 \quad \text{for all } (a,b) \neq (c,d) \in \mathbb{R}^2. \tag{4.2}$$

Let $X = (a - c)$ and $Y = (b - d)$. We now replace the first row in (4.2) with

$$Q(X, Y, c, d) = P(a, b) - P(c, d) - X \frac{\partial}{\partial x} P(c, d) - Y \frac{\partial}{\partial y} P(c, d).$$

By Taylor's Theorem, the coordinates of Q are polynomials in X and Y containing quadratic and cubic terms only and the coefficient with those terms are polynomials in c and d.

Similarly we replace the last row with $\frac{\partial}{\partial x} P(a, b) - \frac{\partial}{\partial x} P(c, d) = \frac{\partial}{\partial X} Q(X, Y, c, d)$ which is a quadratic polynomials with no constant term. The resulting determinant $R(X, Y, c, d)$ is a fifth degree polynomial in X and Y

$$R(X, Y, c, d) = \sum_{3 \leqslant i+j \leqslant 5} A_{i,j}(c, d) X^i Y^j, \tag{4.3}$$

where $A_{i,j}(c, d)$ are polynomials in c and d.

Assumption (4.2) implies that

$$R(X, Y, c, d) \geqslant 0 \quad \text{for all } X, Y \in \mathbb{R}, \text{ and } = 0 \text{ iff } X = Y = 0.$$

Thus (cf. [1, Proposition 6.3.4]) $R(X, Y, c, d)$ is a sum of squares of polynomials. Therefore the coefficients in front of the monomials of odd degree must be equal to zero for all c and d.

Hence

$$A_{i,j}(c, d) = 0 \quad \text{for all } c, d, \text{ and } i, j \text{ such that } i + j = 3, 5.$$

We use Maple to solve the resulting system of equations for $a_{i,j}^{(k)}$. As a result we obtain a parametrized family of solutions. Using Maple once more we verified that for those values of $a_{i,j}^{(k)}$, the equation $A_{0,4}(c,d) \equiv 0$ has a real solution. That means that for some c and d

$$R(X,Y,c,d) = A_{1,3}(c,d)XY^3 + A_{2,2}(c,d)X^2Y^2 + A_{3,1}(c,d)X^3Y + A_{4,0}(c,d)X^4.$$

Thus $R(X,Y,c,d) = 0$ if $X = (a-c) = 0$ and $Y \neq 0$. $\square$

5. Case 5: Interpolation with four derivatives

In this section we settle the last case of Λ-interpolation with Λ consisting of two point evaluations and all first partial derivatives at these points, i.e.

$$\Lambda = \left\{ \delta_u, \delta_v, \delta_u \circ \frac{\partial}{\partial x}, \delta_u \circ \frac{\partial}{\partial y}, \delta_v \circ \frac{\partial}{\partial x}, \delta_v \circ \frac{\partial}{\partial y} \right\}.$$

Namely we will prove the following:

Theorem 10. *The space $\Phi = span[1, x, y, x^2 - y^2, yx, x^3 - 3y^2x, -3x^2y + y^3]$ is Λ-interpolating at any two distinct points u and v. No six-dimensional space Φ has this property.*

Proof. The first part of the statement is a consequence of Theorem 12 below. The last part is a simple application of the "Mairhuber argument" that implies a more general result. $\square$

Theorem 11. *Let F, G and H be arbitrary continuous functions mapping $\mathbb{R}^2$ into $\mathbb{R}^6$. Then for any circle $C \subset \mathbb{R}^2$ there exists a pair of points $u \neq v \in C$ such that the 6×6 determinant*

$$\det \begin{vmatrix} - & F(u) & - \\ - & F(v) & - \\ - & G(u) & - \\ - & G(v) & - \\ - & H(u) & - \\ - & H(v) & - \end{vmatrix} = 0.$$

Proof. Consider the above determinant. As u and v are rotated into each other, three consecutive pairs of rows alternate and hence the sign of the determinant changes. Once again, by the intermediate value theorem, we conclude the existence of u and v for which the above determinant is zero. $\square$

Theorem 12. *For every function $f \in C^1(\mathbb{R}^2)$, the $(4k-1)$-dimensional space Φ spanned by harmonic polynomials of degree $(2k-1)$ interpolates the values of the function and all of its partial derivatives of the first order at any k distinct points in $\mathbb{R}^2$.*

Proof. Consider the k distinct points $u_1, u_2, \ldots, u_k \in \mathbb{R}^2$ as complex numbers $u_j = (x_j, y_j) = x_j + iy_j$. Let $\{\alpha_j, \beta_j, \gamma_j \in \mathbb{R} : j = 1, \ldots, k\}$ be given. Let $p(z) = a_0 + a_1 z + \cdots + a_{2k-1} z^{2k-1} = h(x, y) + ig(x, y)$ be a complex polynomial such that

$$p(u_j) = \alpha_j, \quad p'(u_j) = \beta_j - i\gamma_j.$$

Then $h(u_j) = \alpha_j$ and by the Cauchy–Riemann equation we have

$$p'(u_j) = \frac{\partial}{\partial x}(h(x, y) + ig(x, y)) = \frac{\partial h}{\partial x}(u_j) - i\frac{\partial h}{\partial y}(u_j) = \beta_j - i\gamma_j.$$

Hence h is the harmonic polynomial with the desired property. □

6. Concluding remarks

(1) In this paper we were only concerned with interpolation of the values of a function and its first-order partial derivatives at two points in $\mathbb{R}^2$. Let us mention what little is known about Hermite interpolation at three or more points in $\mathbb{R}^2$ or at two points in $\mathbb{R}^d$, $d > 2$:

Using tools of Differential Topology the following general upper bound was proved in [11]:

Theorem 13. *Let $u_1, u_2, \ldots, u_k \in \mathbb{R}^d$ be an arbitrary collection of k distinct points. For each $j = 1, \ldots, k$ consider a collection of n_j distinct functionals $\Lambda(j) = \{\delta_{u_j} \circ L_1^{(j)}, \ldots, \delta_{u_j} \circ L_{n_j}^{(j)}\}$ where $L_l^{(j)}$ are arbitrary operators on $C^\infty(\mathbb{R}^d)$. Let $\Lambda_k(\bar{n}) = \cup\Lambda(j)$ and let $m = \#\Lambda_k(\bar{n})$, the cardinality of $\Lambda_k(\bar{n})$. Then there exists a subspace $\Phi \subset C(\mathbb{R}^d)$ with $\dim \Phi = dk + m$ that interpolates $\Lambda_k(\bar{n})$ for an arbitrary choice of distinct points $u_1, u_2, \ldots, u_k \in \mathbb{R}^d$.*

Even for this, rather weak estimate, only the existence of a subspace $\Phi \subset C(\mathbb{R}^d)$ with $\dim \Phi = dk + m$ is demonstrated. Harmonic polynomials, that came so handy in Theorem 12, are useless for interpolation of higher derivatives, since the Laplacian of such polynomials is equals to zero.

No reasonable lower bound is known to the author. Some lower bounds for Lagrange interpolation are given in [3,12,14,15]. Yet, the exact values of the minimal dimension of a space that interpolates at five points in $\mathbb{R}^2$ or four points in $\mathbb{R}^3$ are not known.

(2) The “negative results” (Theorems 2 and 10) were proved in greater generality, than necessary. Instead of interpolating an arbitrary function and some of its partial derivatives at two arbitrary points, we in fact obtained estimates for the minimal dimension of a subspace of $C(\mathbb{R}^2)$ needed to interpolate simultaneously some set of continuous functions at two arbitrary points. For instance, Theorem 2 shows that for every $F : \mathbb{R}^2 \to \mathbb{R}^3$ there exists a pair of points $u \neq v \in \mathbb{R}^2$ such that the 3×3 determinant

$$\det \begin{vmatrix} - & F(u) & - \\ - & F(v) & - \\ - & \dfrac{\partial}{\partial x} F(u) & - \end{vmatrix} = 0,$$

by showing that for every two functions $F, G : \mathbb{R}^2 \to \mathbb{R}^3$, there exists a pair of points $u \neq v \in \mathbb{R}^2$ such that the 3×3 determinant

$$\det \begin{vmatrix} - & F(u) & - \\ - & F(v) & - \\ - & G(u) & - \end{vmatrix} = 0.$$

Comparison of Claim 9 with the proposition bellow suggests that these two problems are not equivalent.

Proposition 14. *There exist three continuous functions F, G and $H : \mathbb{R}^2 \to \mathbb{R}^5$, such that the 5×5 determinant*

$$\det \begin{vmatrix} - & F(u) & - \\ - & F(v) & - \\ - & G(v) & - \\ - & H(v) & - \\ - & G(u) & - \end{vmatrix} \neq 0$$

for any $u \neq v \in \mathbb{R}^2$.

Proof. Consider the functions:

$$F(x, y) := (1, 0, 0, x, y);\ G(x, y) := (0, 1, 0, -y, x);\ H(x, y) := (0, 0, 1, 0, 0).$$

The resulting determinant is

$$\det \begin{bmatrix} 1 & 0 & 0 & a & b \\ 1 & 0 & 0 & c & d \\ 0 & 1 & 0 & -d & c \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -b & a \end{bmatrix} = 2ca + 2db - c^2 - d^2 - a^2 - b^2$$
$$= -(a-c)^2 - (b-d)^2. \qquad \square$$

(3) It was observed by one of the referees, that all the polynomial spaces in all the examples are D-invariant (invariant with respect to partial derivatives), and therefore shift invariant. Using this property one can take one interpolation node at the origin, without loss of generality, which would provide simplification in the computation of the appropriate determinants.

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